# A Study on Monotone Self-Dual Boolean Functions

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**Abstract**

This paper shows that monotone self-dual Boolean functions in irredundant disjuntive normal form (IDNF) do not have more variables than disjuncts. Monotone self-dual Boolean functions in IDNF with the same number of variables and disjuncts are examined. An algorithm is proposed to test whether a monotone Boolean function in IDNF with $n$ variables and $n$ disjuncts is self-dual. The runtime of the algorithm is $O(n^4)$.

## 1. Introduction

The problem of testing whether a monotone Boolean function in irredundant disjuntive normal form (IDNF) is self-dual is one of few problems in circuit complexity whose precise tractability status is unknown. This famous problem is called the *monotone self-duality problem* (Eiter et al., 2008). It impinges upon many areas of computer science, such as artificial intelligence, distributed systems, database theory, and hypergraph theory (Makino, 2003; Eiter and Gottlob, 2002).

Consider a monotone Boolean function $f$ in IDNF. Suppose that $f$ has $k$ variables and $n$ disjuncts:

$$f(x_1, x_2, \ldots, x_k) = D_1 \vee D_2 \vee \cdots \vee D_n$$

where each disjunct $D_i$ is a prime implicant of $f$, $i = 1, \ldots n$. The relationship between $k$ and $n$ is a key aspect of the monotone self-duality problem. Prior work has shown that if $f$ is self-dual then $k \leq n^2$ (Fredman and Khachiyan, 1996; Gaur and Krishnamurti, 2008). We improve on this result. In Section 2, by Corollary 1, we show that if $f$ is self-dual then $k \leq n$. In Section 3, we consider the monotone self-duality problem for Boolean functions with the same number of variables and disjuncts (i.e., $n = k$). For such functions, we propose an algorithm that runs in $O(n^4)$ time.

### *1.1. Definitions*

**Definition 1** *Consider $k$ independent* **Boolean variables***, $x_1, x_2, \ldots, x_k$.* **Boolean literals** *are Boolean variables and their complements, i.e., $x_1, \bar{x}_1, x_2, \bar{x}_2, \ldots, x_k, \bar{x}_k$.*

**Definition 2** *A* **disjunct (D)** *of a Boolean function $f$ is an AND of literals, e.g., $D = x_1\bar{x}_3x_4$, that implies $f$.* **A disjunct set (SD)** *is a set containing all the disjunct's literals, e.g., if $D = x_1\bar{x}_3x_4$ then $SD = \{x_1, \bar{x}_3, x_4\}$. A* **disjunctive normal form (DNF)** *is an OR of disjuncts.*

**Definition 3** *A* **prime implicant (PI)** *of a Boolean function $f$ is a disjunct that implies $f$ such that removing any literal from the disjunct results in a new disjunct that does not imply $f$.*

**Definition 4** *An* **irredundant disjunctive normal form (IDNF)** *is a DNF where each disjunct is a PI of a Boolean function $f$ and no PI can be deleted without changing $f$.*

**Definition 5** *Boolean functions $f$ and $g$ are* **dual pairs** *iff $f(x_1, x_2, \ldots, x_k) = g^D = \bar{g}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k)$. A Boolean function $f$ is* **self-dual** *iff $f(x_1, x_2, \ldots, x_k) = f^D = \bar{f}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k)$.*

*Given an expression for a Boolean function in terms of AND, OR, NOT, 0, and 1, its dual can also be obtained by interchanging the AND and OR operations as well as interchanging the constants 0 and 1. For example, if $f(x_1, x_2, x_3) = x_1x_2 \vee \bar{x}_1x_3$ then $f^D(x_1, x_2, x_3) = (x_1 \vee x_2)(\bar{x}_1 \vee x_3)$. A trivial example is that for $f = 1$, the dual is $f^D = 0$.*

**Definition 6** *A Boolean function $f$ is* **monotone** *if it can be constructed using only the AND and OR operations (specifically, if it can constructed without the NOT operation).*

**Definition 7** **The Fano plane** *is the smallest finite projective plane with seven points and seven lines such that and every pair of its lines intersect in one point.* **A Boolean function that represents the Fano plane** *is a monotone self-dual Boolean function with seven variables and seven disjuncts such that every pair of its disjuncts intersect in one variable. An example is $f = x_1x_2x_3 \vee x_1x_4x_5 \vee x_1x_6x_7 \vee x_2x_4x_6 \vee x_2x_5x_7 \vee x_3x_4x_7 \vee x_3x_5x_6$.*

## 2. Number of disjuncts versus number of variables

Our main contribution in this section is Theorem 1. It defines a necessary condition for monotone self-dual Boolean functions. For such functions, there exists a *matching* between its variables and disjuncts, i.e., every variable can be paired to a distinct disjunct that contains the variable. From this theorem we derive our two main results, presented as Corollary 1 and Corollary 2.

### 2.1. Preliminaries

We define the **intersection property** as follows. A Boolean function $f$ satisfies the *intersection property* if every pair of its disjuncts has a non-empty intersection.

**Lemma 1** *(Fredman and Khachiyan, 1996) Consider a monotone Boolean function $f$ in IDNF. If $f$ is self-dual then $f$ satisfies the intersection property.*

**Proof of Lemma 1:** The proof is by contradiction. Consider a disjunct $D$ of $f$. We assign 1's to the all variables of $D$ and 0's to the other variables of $f$. This makes $f = 1$. If $f$ does not satisfy the intersection property then there must be a disjunct of $f$ having all assigned 0's. This makes $f^D = 0$, so $f \neq f^D$. This is a contradiction. □

**Lemma 2** *Consider a monotone Boolean function $f$ in IDNF satisfying the intersection property. Suppose that we obtain a new Boolean function $g$ by removing one or more disjuncts from $f$. There is an assignment of 0's and 1's to the variables of $g$ such that every disjunct of $g$ has both a 0 and a 1.*

**Proof of Lemma 2:** Consider one of the disjuncts that was removed from $f$. We focus on the variables of this disjunct that are also variables of $g$. Suppose that we assign 1's to all of these variables of $g$ and 0's to all of the other variables of $g$. Since $f$ is in IDNF, the assigned 1's do not make $g = 1$. Therefore $g = 0$; every disjunct of $g$ has at least one assigned 0. Since $f$ satisfies the intersection property, every disjunct of $g$ has at least one assigned 1. As a result, every disjunct of $g$ has both a 0 and a 1. □

We define a **matching** between a variable $x$ and a disjunct $D$ as follows. There is a matching between $x$ and $D$ iff $x$ is a variable of $D$. For example, if $D = x_1x_2$ then there is a matching between $x_1$ and $D$ as well as $x_2$ and $D$.

**Lemma 3** *Consider a monotone Boolean function $f$ in IDNF satisfying the intersection property. Suppose that $f$ has $k$ variables and $n$ disjuncts. If each of the $b$ variables of $f$ can be matched with a distinct disjunct of $f$ where $b < k$ and $b < n$, and all other unmatched disjuncts of $f$ do not have any of the matched variables, then $f$ is not self-dual.*

**Proof of Lemma 3:** Lemma 3 is illustrated in Table 1. Note that a variable $x_i$ is matched with a disjunct $D_i$ for every $i = 1, \ldots, b$. To prove that $f$ is not self-dual, we assign 0's and 1's to the variables of $f$ such that every disjunct of $f$ has both 0 and 1. This results in $f = 0$ and $f^D = 1$; $f \neq f^D$. We first assign 0's and 1's to the variables of $D_{b+1} \vee \ldots \vee D_n$ to make each disjunct of $D_{b+1} \vee \ldots \vee D_n$ have both a 0 and a 1. Lemma 2 allows us to do so. Note that none of the variables $x_1, \ldots, x_b$ has an assignment yet. Since $f$ satisfies the intersection property, each disjunct of $D_1 \vee \ldots \vee D_b$ should have at least one previously assigned 0 or 1. If a disjunct of $D_1 \vee \ldots \vee D_b$ has a previously assigned 1 then we assign 0 to its matched (circled) variable; if a disjunct of $D_1 \vee \ldots \vee D_b$ has a previously assigned 0 then we assign 1 to its matched (circled) variable. As a result, every disjunct of $f$ has both a 0 and a 1; therefore $f$ is not self-dual.

| $D_1$ | $D_2$ | ......... | $D_{b-1}$ | $D_b$ | $D_{b+1}$..............$D_n$ |
|---|---|---|---|---|---|
| . | . | ......... | . | . | .................. |
| . | . | ......... | . | . | .................. |
| ($x_1$) | ($x_2$) | ......... | ($x_{b-1}$) | ($x_b$) | .................. |
| | | | | | no $x_1, \ldots\ldots, x_b$ |

Table 1: An illustration of Lemma 3.

□

**Lemma 4** *(Fredman and Khachiyan, 1996) Boolean functions $f$ and $g$ are dual pairs iff a Boolean function $af \vee bg \vee ab$ is self-dual where $a$ and $b$ are Boolean variables.*

**Proof of Lemma 4:** From the definition of duality, if $af \vee bg \vee ab$ is self-dual then $(af \vee bg \vee ab)_{a=1,\, b=0} = f$ and $(af \vee bg \vee ab)_{a=0,\, b=1} = g$ are dual pairs. From the definition of duality, if $f$ and $g$ are dual pairs then $(af \vee bg \vee ab)^D = (a^D \vee f^D)(b^D \vee g^D)(a^D \vee b^D) = (a \vee g)(b \vee f)(a \vee b) = (af \vee bg \vee ab)$.

□

### *2.2. The Theorem*

**Theorem 1** *Consider a monotone Boolean function $f$ in IDNF. If $f$ is self-dual then each variable of $f$ can be matched with a distinct disjunct.*

Before proving the theorem we elucidate it with examples.

**Example 1** *Consider a monotone self-dual Boolean function in IDNF*

$$f = x_1x_2 \vee x_1x_3 \vee x_2x_3.$$

*The function has three variables $x_1$, $x_2$, and $x_3$, and three disjuncts $D_1 = x_1x_2$, $D_2 = x_1x_3$, and $D_3 = x_2x_3$. As shown in Table 2, every variable is matched with a distinct disjunct; the circled $x_1$, $x_2$, and $x_3$ are matched with $D_1$, $D_3$, and $D_2$, respectively. We see that the theorem holds for this example. Note that the required matching – each variable to a distinct disjunct – might not be unique. For this example, another possibility is having $x_1$, $x_2$, and $x_3$ matched with $D_2$, $D_1$, and $D_3$, respectively.*

| $D_1$ | $D_3$ | $D_2$ |
|---|---|---|
| $x_2$ | $x_3$ | $x_1$ |
| ($x_1$) | ($x_2$) | ($x_3$) |

Table 2: An example to illustrate Theorem 1.

**Example 2** *Consider a monotone self-dual Boolean function in IDNF*

$$f = x_1x_2x_3 \vee x_1x_3x_4 \vee x_1x_5x_6 \vee x_2x_3x_6 \vee x_2x_4x_5 \vee x_3x_4x_6 \vee x_3x_5.$$

*The function has six variables $x_1$, $x_2$, $x_3$, $x_4$, $x_5$, and $x_6$, and seven disjuncts $D_1 = x_1x_2x_3$, $D_2 = x_1x_3x_4$, $D_3 = x_1x_5x_6$, $D_4 = x_2x_3x_6$, $D_5 = x_2x_4x_5$, $D_6 = x_3x_4x_6$, and $D_7 = x_3x_5$. As shown in Table 3, every variable is matched with a distinct disjunct; the circled $x_1$, $x_2$, $x_3$, $x_4$, $x_5$, and $x_6$ are matched with $D_1$, $D_4$, $D_2$, $D_5$, $D_3$, and $D_6$, respectively. We see that the theorem holds for this example.*

| $D_1$ | $D_4$ | $D_2$ | $D_5$ | $D_3$ | $D_6$ | $D_7$ |
|---|---|---|---|---|---|---|
| $x_2$ | $x_3$ | $x_1$ | $x_2$ | $x_1$ | $x_3$ | |
| $x_3$ | $x_6$ | $x_4$ | $x_5$ | $x_6$ | $x_4$ | $x_3$ |
| ($x_1$) | ($x_2$) | ($x_3$) | ($x_4$) | ($x_5$) | ($x_6$) | $x_5$ |

Table 3: An example to illustrate Theorem 1.

**Proof of Theorem 1:** The proof is by contradiction. We suppose that at most $a$ variables of $f$ can be matched with distinct disjuncts, where $a < k$. We consider two cases, $n = a$ and $n > a$ where $n$ is the number of disjuncts of $f$. For both cases, we find an assignment of 0's and 1's to the variables of $f$ such that every disjunct of $f$ has both a 0 and a 1. This results in a contradiction since such an assignment makes $f = 0$ and $f^D = 1$; $f \neq f^D$.

**Case 1:** $n = a$.

This case is illustrated in Table 4. To make every disjunct of $f$ have both a 0 and a 1, we first assign 0 to $x_1$ and 1 to $x_{a+1}$. Then we assign a 0 or a 1 to each of the variables $x_2, \ldots, x_a$ step by step. In each step, if a disjunct has a previously assigned 1 then we assign 0 to its matched (circled) variable; if a disjunct has a previously assigned 0 then we assign 1 to its matched (circled) variable. After these steps, if every disjunct of $f$ has both a 0 and a 1 then we have proved that $f$ is not self-dual. If there remain disjuncts, these disjuncts should not have any previously assigned variables. Lemma 3 identifies this condition and it tells us that $f$ is not self-dual. This is a contradiction.

| $D_1$ | $D_2$ | ......... | $D_{n-1}$ | $D_n$ |
|---|---|---|---|---|
| . | . | ......... | . | . |
| $x_{a+1}$ | . | ......... | . | . |
| ($x_1$) | ($x_2$) | ......... | ($x_{a-1}$) | ($x_a$) |

Table 4: An illustration of Case 1.

**Case 2:** $n > a$

This case is illustrated in Table 5. We show that $f$ always satisfies the condition in Lemma 3; accordingly $f$ is not self-dual.

As shown in Table 5, the expression $D_{a+1} \vee \ldots \vee D_n$ does not have the variable $x_1$ or the variable $x_{a+1}$. If it had then at least $a + 1$ variables would be matched; this would go against our assumption.

For example, if $D_{a+1} \vee \ldots \vee D_n$ has $x_1$ then $x_1$ would be matched with a disjunct from $D_{a+1} \vee \ldots \vee D_n$ and $x_{a+1}$ would be matched with $D_1$. So $a + 1$ variables would be matched with distinct disjuncts.

| $D_1$ | $D_2$ | ........ | $D_{a-1}$ | $D_a$ | $D_{a+1}$ ............... $D_n$ |
|---|---|---|---|---|---|
| . | . | ......... | . | . | .................. |
| $x_{a+1}$ | . | ......... | . | . | .................. |
| ($x_1$) | ($x_2$) | ......... | ($x_{a-1}$) | ($x_a$) | .................. |
| | | | | | no $x_1$ no $x_{a+1}$ |

Table 5: An illustration of Case 2.

If $D_{a+1} \vee \ldots \vee D_n$ does not have any of the variables $x_2, \ldots, x_a$ then $f$ satisfies the condition in Lemma 3; $f$ is not self-dual. If it does then the number of disjuncts not having $x_1$ or $x_{a+1}$ increases. This is illustrated in Table 6. Suppose that $D_{a+1} \vee \ldots \vee D_n$ has variables $x_j, \ldots, x_{a-1}$ where $j \geq 2$. As shown in the table, $D_j \vee \ldots \vee D_n$ does not have $x_1$ or $x_{a+1}$. If it had then at least $a + 1$ variables would be matched; this would go against our assumption. For example, if $D_j$ had $x_{a+1}$ then $x_{a+1}$ would be matched with $D_j$ and $x_j$ would be matched with a disjunct from $D_{a+1} \vee \ldots \vee D_n$. So $a + 1$ variables would be matched with distinct disjuncts.

| $D_1$ | $D_2$ | ...... | $D_{j-1}$ | $D_j$ | ...... | $D_{a-1}$ | $D_a$ | $D_{a+1}$ ........... $D_n$ |
|---|---|---|---|---|---|---|---|---|
| . | . | ...... | . | . | ...... | . | . | .................. |
| $x_{a+1}$ | . | ...... | . | . | ...... | . | . | .................. |
| ($x_1$) | ($x_2$) | ...... | ($x_{j-1}$) | ($x_j$) | ...... | ($x_{a-1}$) | ($x_a$) | .................. |
| | | | | no $x_1$ no $x_{a+1}$ | | | | |

Table 6: An illustration of Case 2.

If $D_j \vee \ldots \vee D_n$ does not have any of the variables $x_2, \ldots, x_{j-1}$ then $f$ satisfies Lemma 3; $f$ is not self-dual. If it does have any of these variables then the number of disjuncts not having $x_1$ or $x_{a+1}$ increases.

As a result the number of disjuncts not having $x_1$ or $x_{a+1}$ increases unless the condition in Lemma 3 is satisfied. Since there must be disjuncts having $x_1$ or $x_{a+1}$, this increase should eventually stop. When it stops, the condition in Lemma 3 will be satisfied. As a result, $f$ is not self-dual. This is a contradiction. □

**Corollary 1** *Consider a monotone Boolean function $f$ in IDNF. Suppose that $f$ has $k$ variables and $n$ disjuncts. If $f$ is self-dual then $k \leq n$.*

**Proof of Corollary 1:** We know that if $f$ is self-dual then $f$ should satisfy the matching defined in Theorem 1. This matching requires that $f$ does not have more variables than disjuncts, so $k \leq n$. □

**Corollary 2** *Consider monotone Boolean functions $f$ and $g$ in IDNF. Suppose that $f$ has $k$ variables and $n$ disjuncts and $g$ has $k$ variables and $m$ disjuncts. If $f$ and $g$ are dual pairs then $k \leq n + m - 1$.*

**Proof of Corollary 2:**

From Lemma 4 we know that the Boolean functions $f$ and $g$ are dual pairs iff a Boolean function $af \vee bg \vee ab$ is self-dual where $a$ and $b$ are Boolean variables. If neither $a$ nor $b$ is a variable of $f$ (or of $g$) then $af \vee bg \vee ab$ has $n + m + 1$ disjuncts and $k + 2$ variables. From Corollary 1, we know that $k + 2 \leq n + m + 1$, so $k \leq n + m - 1$. □

## 3. The self-duality problem

In this section we propose an algorithm to test whether a monotone Boolean function in IDNF with $n$ variables and $n$ disjuncts is self-dual. The runtime of the algorithm is $O(n^4)$.

### 3.1. Preliminaries

**Theorem 2** *(Altun and Riedel, 2010, 2011) Consider a disjunct $D_i$ of a monotone self-dual Boolean function $f$ in IDNF. For any variable $x$ of $D_i$ there exists at least one disjunct $D_j$ of $f$ such that $SD_i \cap SD_j = \{x\}$.*

Before proving the theorem we elucidate it with an example.

**Example 3** *Consider a monotone self-dual Boolean function function in IDNF*

$$f = x_1x_2x_3 \vee x_1x_3x_4 \vee x_1x_5x_6 \vee x_2x_3x_6 \vee x_2x_4x_5 \vee x_3x_4x_6 \vee x_3x_5.$$

*The function has seven disjuncts $D_1 = x_1x_2x_3$, $D_2 = x_1x_3x_4$, $D_3 = x_1x_5x_6$, $D_4 = x_2x_3x_6$, $D_5 = x_2x_4x_5$, $D_6 = x_3x_4x_6$, and $D_7 = x_3x_5$. Consider the disjunct $D_1 = x_1x_2x_3$. Since $SD_1 \cap SD_3 = \{x_1\}$, $SD_1 \cap SD_5 = \{x_2\}$, and $SD_1 \cap SD_6 = \{x_3\}$, the theorem holds for any variable of $D_1$. Consider the disjunct $D_2 = x_1x_3x_4$. Since $SD_2 \cap SD_3 = \{x_1\}$, $SD_2 \cap SD_4 = \{x_3\}$, and $SD_2 \cap SD_5 = \{x_4\}$, the theorem holds for any variable of $D_2$.*

**Proof of Theorem 2:** The proof is by contradiction. Suppose that there is no disjunct $D_j$ of $f$ such that $SD_i \cap SD_j = \{x\}$. From Lemma 1, we know that $D_i$ has a non-empty intersection with every disjunct of $f$. If we extract $x$ from $D_i$ then a new disjunct $D_i'$ should also have a non-empty intersection with every disjunct of $f$. This means that if we assign 1's to the all variables of $D_i'$ then these assigned 1's make $f = f^D = (1 + \ldots)(1 + \ldots)\ldots(1 + \ldots) = 1$. So $D_i'$ implies $f$; $D_i'$ is a disjunct of $f$. This disjunct covers $D_i$. However, in IDNF, all disjuncts including $D_i$ are irredundant, not covered by another disjunct of $f$. So we have a contradiction □

**Lemma 5** *Consider a disjunct $D$ of a monotone self-dual Boolean function $f$ in IDNF. Consider all disjuncts $D_1, \ldots, D_y$ of $f$ such that $SD \cap SD_i = \{x\}$ for every $i = 1, \ldots, y$. A Boolean function $g = (D_{x=1})((D_1 \vee \ldots \vee D_y)_{x=1})^D$ implies (i.e., is covered by) $f$.*

Before proving the lemma we elucidate it with an example.

**Example 4** *Consider a monotone self-dual Boolean function function in IDNF*

$$f = x_1x_2x_3 \vee x_1x_3x_4 \vee x_1x_5x_6 \vee x_2x_3x_6 \vee x_2x_4x_5 \vee x_3x_4x_6 \vee x_3x_5.$$

*The function has seven disjuncts $D_1 = x_1x_2x_3$, $D_2 = x_1x_3x_4$, $D_3 = x_1x_5x_6$, $D_4 = x_2x_3x_6$, $D_5 = x_2x_4x_5$, $D_6 = x_3x_4x_6$, and $D_7 = x_3x_5$. Consider the disjunct $D_1 = x_1x_2x_3$. The disjunct $D_3 = x_1x_5x_6$ is the only disjunct that intersects $D_1$ in $x_1$. Since $g = \left((D_1)_{x_1=1}\right)\left((D_3)_{x_1=1}\right)^D = x_2x_3x_5 \vee x_2x_3x_6$ implies $f$, the lemma holds for this case. The disjuncts $D_6 = x_3x_4x_6$ and $D_7 = x_3x_5$ are the only disjuncts that intersect $D_1$ in $x_3$. Since $g = \left((D_1)_{x_3=1}\right)\left((D_6 \vee D_7)_{x_1=1}\right)^D = x_1x_2x_4x_5 \vee x_1x_2x_5x_6$ implies $f$, the lemma holds for this case.*

**Proof of Lemma 5:** To prove the statement we check if $g = 1$ always makes $f = f^D = 1$ (by assigning 1's to the variables of $g$). Suppose that $f$ has $n$ disjuncts $D_1, \ldots, D_y, D, D_{y+2}, \ldots, D_n$. If $g = 1$ then both $(D_{x=1}) = 1$ and $((D_1 \vee \ldots \vee D_y)_{x=1})^D = 1$. From Lemma 1, we know that if $(D_{x=1}) = 1$ then every disjunct of $D_{y+2}, \ldots \vee, D_n$ has at least one assigned 1. From the definition of duality, we know that if $((D_1 \vee \ldots \vee D_y)_{x=1})^D = 1$ then every disjunct of $D_1, \ldots, D_y$ has at least one assigned 1. As a result, every disjunct of $f$ has at least one assigned 1 making $f = f^D = (1 + \ldots) \ldots (1 + \ldots) = 1$. □

**Lemma 6** *Consider a monotone self-dual Boolean function $f$ in IDNF with $k$ variables. A set of $b$ variables of $f$ has a non-empty intersection with at least $b + 1$ disjunct sets of $f$ where $b < k$.*

Before proving the lemma we elucidate it with an example.

**Example 5** *Consider a monotone self-dual Boolean function function in IDNF*

$$f = x_1x_2x_3x_4 \vee x_1x_5 \vee x_1x_6 \vee x_2x_5x_6 \vee x_3x_5x_6 \vee x_4x_5x_6.$$

*The function has six disjuncts $D_1 = x_1x_2x_3x_4$, $D_2 = x_1x_5$, $D_3 = x_1x_6$, $D_4 = x_2x_5x_6$, $D_5 = x_3x_5x_6$, and $D_6 = x_4x_5x_6$. Consider a set of two variables $\{x_2, x_3\}$; $b = 2$. Since it has a non-empty intersection with three disjunct sets $SD_1$, $SD_4$, and $SD_5$, the lemma holds for this case. Consider a set of one variable $\{x_1\}$; $b = 1$. Since has a non-empty intersection with three disjunct sets $SD_1$, $SD_2$, and $SD_3$, the lemma holds for this case.*

**Proof of Lemma 6:** The proof is by contradiction. From Theorem 1, we know that each of the $k$ variables should be matched with a distinct disjunct, so a set of $b$ variables of $f$ should have a non-empty intersection with at least $b$ disjunct sets of $f$. Suppose that a set of $b$ variables of $f$ has a non-empty intersection with exactly $b$ disjunct sets of $f$. Lemma 3 identifies this condition and it tells us that $f$ is not self-dual. This is a contradiction. □

**Theorem 3** *Consider a monotone self-dual Boolean function $f$ in IDNF with $k$ variables. If every variable of $f$ occurs at least three times then a set of $b$ variables of $f$ has a non-empty intersection with at least $b + 2$ disjunct sets of $f$ where $b < k - 1$.*

**Proof of Theorem 3:** The proof is by induction on $b$.
**The base case:** $b = 1$.

Since a variable of $f$ occurs three times, a set of one variable should have a non-empty intersection with at least three disjunct sets of $f$.

**The inductive step:** Assume that the theorem holds for $b \leq m$ where $m \geq 2$. We show that it also holds for $b = m + 1$.

Consider a set of $m+1$ variables $S = \{x_1, \ldots, x_{m+1}\}$. Consider a disjunct $D$ of $f$ such that $SD \cap S = \{x_1, \ldots, x_c\}$. From Theorem 2, we know that there is at least one disjunct that intersects $D$ in $x_i$ for every $i = 1, \ldots, c$. We consider two cases.

For the cases we suppose that $f$ does not have a disjunct set intersecting $S$ in one variable; if it does then the theorem holds for $S$ (by using the inductive assumption). Also we suppose that $f$ does not have a disjunct set that is a subset of $S$; if it does then it is obvious that the theorem holds for $S$.

**Case 1:** There is only one disjunct that intersects $D$ in $x_i$ for every $i = 1, \ldots, c$.

Suppose that $D_i$ is the only disjunct that intersects $D$ in $x_i$ for every $i = 1, \ldots, c$. Consider a variable set $SD_{x_1-x_c}$ of $((D_1)_{x_1=1} \vee \ldots \vee (D_c)_{x_c=1})$; $SD_{x_1-x_c}$ includes all variables of $((D_1)_{x_1=1} \vee \ldots \vee (D_c)_{x_c=1})$. From Lemma 5, we know that $((D)_{x_i=1})((D_i)_{x_i=1})^D$ implies $f$ for every $i = 1, \ldots, c$. This means that

$f$ should have at least $|SD_{x_1-x_c} \cap S|$ disjuncts such that each of them has one distinct variable from $SD_{x_1-x_c} \cap S = \{x_{c+1}, x_{c+2}, \ldots, x_{m+1}\}$ and none of them is covered by $(D \vee D_1 \vee \ldots \vee D_c)$.

If $SD_{x_1-x_c} \cap S = \{x_{c+1}, x_{c+2}, \ldots, x_{m+1}\}$ then $f$ has at least $|SD_{x_1-x_c} \cap S| = m - c + 1$ disjunct sets such that each of them intersects $\{x_{c+1}, x_{c+2}, \ldots, x_{m+1}\}$ in one variable. Therefore, including $SD$, $SD_1$, $SD_2$, …, and $SD_c$, $f$ has at least $m + 2$ disjunct sets such that each of them has a non-empty intersection with $S$. If $f$ has exactly $m + 2$ disjunct sets then each disjunct of $f$ has a non-empty intersection with $(x_{c+1}x_{c+2} \ldots x_{m+1})(D_{x_1=1,\ldots,x_c=1})$. This means that $f$ should have a disjunct that covers $(x_{c+1}x_{c+2} \ldots x_{m+1})(D_{x_1=1,\ldots,x_c=1})$. Since none of the $m$+2 disjuncts covers $(x_{c+1}x_{c+2} \ldots x_{m+1})(D_{x_1=1,\ldots,x_c=1})$, $f$ needs one more disjunct to cover $(x_{c+1}x_{c+2} \ldots x_{m+1})(D_{x_1=1,\ldots,x_c=1})$ that has a non-empty intersection with $S$. This is a contradiction. As a result, $f$ has at least $m + 3$ disjunct sets such that each of them has a non-empty intersection with $S$; the theorem holds for $S$.

If $SD_{x_1-x_c} \cap S = \{x_{c+1}, x_{c+2}, \ldots, x_n\}$ where $n < m + 1$ then from our inductive assumption we know that the variable set $\{x_{n+1}, x_{n+2}, \ldots, x_{m+1}\}$ intersects at least $m - n + 3$ disjunct sets. As a result, $f$ has at least $(c + 1) + |SD_{x_1-x_c} \cap S| = (n - c) + (m - n + 3) = m + 4$ disjunct sets such that each of them has a non-empty intersection with $S$. So the theorem holds for $S$.

**Case 2:** For at least one of the variables of $x_1, \ldots, x_c$, say $x_c$, there are at least two disjuncts such that each of them intersects $D$ in $x_c$.

The proof has $c$ steps. In each step, we consider all disjuncts of $f$ such that each of them intersects $D$ in $x_i$ where $1 \leq i \leq c$. We first consider disjuncts $D_1, \ldots, D_y$ such that each of them intersects $D$ in $x_1$. Consider a variable set $SD_{x_1}$ of $(D_1 \vee \ldots \vee D_y)_{x_1=1}$; $SD_{x_1}$ includes all variables of $(D_1 \vee \ldots \vee D_y)_{x_1=1}$. From Lemma 5, we know that $(D_{x_1=1})((D_1 \vee \ldots \vee D_y)_{x_1=1})^D$ implies $f$. Therefore along with $D_1 \vee \ldots \vee D_y$, $f$ should have disjuncts that cover $(D_{x_1=1})((D_1 \vee \ldots \vee D_y)_{x_1=1})^D$. This means that $f$ includes a dual-pair of $(D_1 \vee \ldots \vee D_y)_{x_1=1}$ and $((D_1 \vee \ldots \vee D_y)_{x_1=1})^D$. From Lemma 4 and Lemma 6, we know that $SD_{x_1} \cap S$ requires at least $|SD_{x_1} \cap S| + 1$ disjunct sets of $f$ such that each of them has a non-empty intersection with $S$.

We apply the same method for $x_2$, $x_3$, and $x_{c-1}$, respectively. Consider a variable set $SD_{x_i}$ for every $i = 2, \ldots, c - 1$; $SD_{x_i}$ is obtained in the same way as $SD_{x_1}$ was obtained in the first step. In each step if $SD_{x_i} \cap S$ has new variables that are the variables not included in $(SD_{x_1} \cup \ldots \cup SD_{x_{i-1}}) \cap S$, then these new variables result in new disjuncts. From Lemma 4 and Lemma 6, we know that the number of new disjuncts is at least one more than the number of the new variables. Therefore before the last step, including $SD$, $f$ has at least $|(SD_{x_1} \cup \ldots \cup SD_{x_{c-1}}) \cap S| + (c - 1) + 1$ disjunct sets (+1 is for $SD$) such that each of them has a non-empty intersection with $S$.

The last step corresponds to $x_c$. If $|(SD_{x_1} \cup \ldots \cup SD_{x_{c-1}}) \cap S| = ((m + 1) - c)$ then $SD_{x_c}$ does not have any new variables. Since there are at least two disjuncts such that each of them intersects $D$ in $x_c$, $f$ has at least $(m + 1 - c) + (c) + (2) = m + 3$ disjunct sets such that each of them has a non-empty intersection with $S$. So the theorem holds for $S$. If $|(SD_{x_1} \cup \ldots \cup SD_{x_{c-1}}) \cap S| = n$ where $n < (m + 1) - c$ then $S$ has $(m - n - c + 1)$ variables that are not included in $((SD_{x_1} \cup \ldots \cup SD_{x_{c-1}}) \cup SD)$. From our inductive assumption, we know that these $(m - n - c + 1)$ variables results in at least $(m - n - c + 1 + 2)$ new disjunct sets. As a result, $f$ has at least $(m + 1 - c) + (c) + (2) = m + 3$ disjunct sets such that each of them has a non-empty intersection with $S$. So the theorem holds for $S$. □

**Lemma 7** *Consider a monotone self-dual Boolean function $f$ in IDNF with the same number of variables and disjuncts. If $f$ has a variable occurring two times then $f$ has at least two disjuncts of size two.*

**Proof of Lemma 7:** If a variable of $f$, say $x_1$, occurs two times then from Theorem 2, we know that two disjuncts that have $x_1$ should intersect in $x_1$. Consider the disjuncts $x_1x_{a1} \ldots x_{an}$ and $x_1x_{b1} \ldots x_{bm}$ of $f$.

From Lemma 5, we know that both $g = (x_{a1} \dots x_{an})(x_{b1} \vee \dots \vee x_{bm})$ and $h = (x_{b1} \dots x_{bm})(x_{a1} \vee \dots \vee x_{an})$ should be covered by $f$. Note that $g$ and $h$ have total of $n+m$ disjuncts. These $n+m$ disjuncts should be covered by at most $n+m-2$ disjuncts of $f$; otherwise Lemma 6 is violated. For example, if $n+m$ disjuncts are covered by $n+m-1$ disjuncts of $f$ then along with the disjuncts $x_1 x_{a1} \dots x_{an}$ and $x_1 x_{b1} \dots x_{bm}$ there are $n+m+1$ disjuncts having $n+m+1$ variables. This means that a set of the remaining variables, say $b$ variables, has a non-empty intersection with at most $b$ disjuncts of $f$, so Lemma 6 is violated.

Any disjunct of $f$ with more than two variables can only cover one of the $m+n$ disjuncts of $g \vee h$. Therefore to cover $m+n$ disjuncts of $g \vee h$ with $m+n-2$ disjuncts, $f$ needs disjuncts of size two. Since a disjunct of size two can cover at most two of the $m+n$ disjuncts of $g \vee h$, $f$ should have at least two disjuncts of size two. □

**Lemma 8** *Consider a monotone self-dual Boolean function $f$ in IDNF with the same number of variables and disjuncts. If each variable of $f$ occurs at least three times then $f$ is a unique Boolean function that represents the Fano plane.*

**Proof of Lemma 8:** We consider two cases.
**Case 1:** A pair of disjuncts of $f$ intersect in multiple variables.

We show that if a pair of disjuncts of $f$ intersect in multiple variables then $f$ is not self-dual. Consider two disjuncts $D_1$ and $D_2$ of $f$ such that they intersect in multiple variables. Suppose that both $D_1$ and $D_2$ have variables $x_1$ and $x_2$. This case is illustrated in Table 7. Note that $x_3$, $x_4$, …, $x_k$ are matched with $D_3$, $D_4$, …, $D_k$, respectively. This is called *perfect matching*. Hall's theorem describes a necessary and sufficient condition for this matching: a subset of $b$ variables of $\{x_3, \dots, x_k\}$ has a non-empty intersection with at least $b$ disjunct sets from $SD_3, \dots, SD_k$. From Theorem 3, we know that a set of $b$ variables of $f$ has a non-empty intersection with at least $b+2$ disjunct sets of $f$. This satisfies the necessary and sufficient condition for the perfect matching between $x_3, \dots, x_k$ and $D_3, \dots, D_k$.

We find an assignment of 0's and 1's to the variables of $f$ such that every disjunct of $f$ has both a 0 and a 1. To make every disjunct of $f$ have both 0 and 1, we first assign 0 to $x_1$ and 1 to $x_2$. Then we assign a 0 or a 1 to each of the variables $x_3, \dots, x_k$ step by step. In each step, if a disjunct has a previously assigned 1 then we assign 0 to its matched (circled) variable; if a disjunct has a previously assigned 0 then we assign 1 to its matched (circled) variable. After these steps, if every disjunct of $f$ has both a 0 and a 1 then we have proved that $f$ is not self-dual. If there remain disjuncts, these disjuncts should not have any previously assigned variables. Lemma 3 identifies this condition and it tells us that $f$ is not self-dual.

| $D_1$ | $D_2$ | $D_3$ | ......... | $D_{n-1}$ | $D_k$ |
|---|---|---|---|---|---|
| . | . | . | ......... | . | . |
| $x_2$ | $x_1$ | . | ......... | . | . |
| $x_1$ | $x_2$ | $x_3$ | ......... | $x_{k-1}$ | $x_k$ |

Table 7: An illustration of Case 1.

**Case 2:** Every pair of disjuncts of $f$ intersect in one variable.

Suppose that a variable of $f$, say $x_1$, occurs three times. Consider disjuncts $D_1 = x_1 x_{a1} \dots x_{an}$, $D_2 = x_1 x_{b1} \dots x_{bm}$, and $D_3 = x_1 x_{c1} \dots x_{cl}$ of $f$ where $n \leq m \leq l$. From Lemma 5, we know that $f$ should cover $(x_{a1} \dots x_{an})(x_{b1} \vee \dots \vee x_{bm})(x_{c1} \vee \dots \vee x_{cl})$ where $n \leq m \leq l$. This means that $f$ should cover $m \cdot l$ disjuncts. These disjuncts are covered by at least $m \cdot l$ disjuncts of $f$; otherwise the intersection property does not hold for $f$. Along with $D_1$, $D_2$, and $D_3$, $f$ has $m \cdot l + 3$ disjuncts having $m + n + l + 1$ variables. From Lemma 1, we know that $m \cdot l + 3 \leq m + n + l + 1$. The only solution of this inequality is that $n = 2$, $m = 2$, and $l = 2$. This results in a self-dual Boolean function representing the Fano plane, e.g., $f = x_1x_2x_3 \vee x_1x_4x_5 \vee x_1x_6x_7 \vee x_2x_4x_6 \vee x_2x_5x_7 \vee x_3x_4x_7 \vee x_3x_5x_6$.

If a variable of $f$ occurs more than three times then the value on left hand side of the inequality $m \cdot l + 3 \leq m + n + l$ increases more than that on the right hand side does, so there is no solution. □

**Lemma 9** *A Boolean function $f$ is self-dual iff $f_{x_a=x_b}$, $f_{x_a=x_c}$, and $f_{x_b=x_c}$ are all self-dual Boolean functions where $x_a$, $x_b$, and $x_c$ are any three variables of $f$.*

**Proof of Lemma 9:** From the definition of duality, $f$ is self-dual iff each assignment of 0's and 1's to the variables of $f$, corresponding to a row of the truth table, satisfies $f(x_1, x_2, \ldots, x_k) = \bar{f}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k)$. Any dependency between variables of $f$ only eliminates some rows of $f$'s truth table. Therefore, if $f$ is self-dual then $f_{x_a=x_b}$, $f_{x_a=x_c}$, and $f_{x_b=x_c}$ are all self-dual. For each row of $f$'s truth table either $x_a = x_b$ or $x_a = x_c$, or $x_b = x_c$. Therefore, if $f_{x_a=x_b}$, $f_{x_a=x_c}$, and $f_{x_b=x_c}$ are all self-dual then $f$ is self-dual. □

### *3.2. The Algorithm*

We present a four-step algorithm:

**Input:** A monotone Boolean function $f$ in IDNF with $n$ variables $n$ disjuncts.

**Output:** "YES" if $f$ is self-dual; "NO" otherwise.

1. Check if $f$ is a single variable Boolean function. If it is then return "YES".
2. Check if $f$ represents the Fano plane. If it does then return "YES".
3. Check if the intersection property holds for $f$. If it does not then return "NO".
4. Check if $f$ has two disjuncts of size two, $x_ax_b$ and $x_ax_c$ where $x_a$, $x_b$, and $x_c$ are variables of $f$. If it does not then return "NO"; otherwise obtain a new function $f = f_{x_b=x_c}$ in IDNF. Repeat this step until $f$ consists of a single variable; in this case, return "YES".

If $f$ is self-dual then $f$ should be in one of the following three categories: (1) $f$ is a single variable Boolean function; (2) at least one variable of $f$ occurs two times; (3) each variable of $f$ occurs at least three times. From Theorem 2, we know that if $f$ is self-dual and not in (1) then every variable of $f$ should occur at least two times, so $f$ should be in either (2) or (3). Therefore these three categories cover all possible self-dual Boolean functions.

The first step of our algorithm checks if $f$ is self-dual and in (1). The second step of our algorithm checks if $f$ is self-dual and in (3). From Lemma 8, we know that if $f$ is self-dual and in (3) then $f$ is a unique Boolean function that represents the Fano plane. The third and fourth steps of our algorithm check if $f$ is self-dual and in (2). From Lemma 1, we know that if $f$ is self-dual then $f$ should satisfy the intersection property. From Lemma 7, we know that if $f$ is self-dual and in (2) then $f$ should have at least two disjuncts of size two, $x_ax_b$ and $x_ax_c$. From Lemma 9, we know that $f$ is self-dual iff $f_{x_a=x_b}$, $f_{x_a=x_c}$, and $f_{x_b=x_c}$ are all self-dual. Since $f$ satisfies the intersection property, both $f_{x_a=x_b} = x_a$ and $f_{x_a=x_c} = x_a$ are self-dual. This means that $f$ is self-dual iff $f_{x_b=x_c}$ is self-dual. Note that $f_{x_b=x_c}$ in IDNF has $n - 1$ variables and $n - 1$ disjuncts. Since $f_{x_b=x_c}$ satisfies the intersection property and does not represent the Fano plane, we just need to repeat step four to check if the function is self-dual. Note that to check if $f$ is self-dual and in (2), we need to repeat step four at most $n$ times.

The steps three and four of the algorithm run in $O(n^4)$ and $O(n^3)$ time, respectively. Therefore the run time of the algorithm is that of the step three $O(n^4)$.

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